Analytic and numerical analysis of the longitudinal coupling impedance of a rectangular slot in a thin coaxial liner

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Beam pipes of high-energy superconducting colliders require a shielding tube (liner) with pumping slots to screen cold chamber walls from synchrotron radiation. Pumping slots in the liner walls are required to keep high vacuum inside the beam pipe and provide for a long beam lifetime. As previously discussed [Fedotov and Gluckstern, Phys. Rev. E **54**, 1930 (1996)], for a long narrow slot whose length may be comparable with the wavelength, the usual static approximation for the polarizability and susceptibility that enter into the impedance is a poor one. Therefore, finding semianalytic expressions for the impedance of a rectangular slot in a broad frequency range is highly desirable. We develop a general analysis based on a variational formulation, which includes both the realistic coaxial structure of the beam-pipe and the effect of finite wavelength, in order to calculate the coupling impedance of a rectangular slot in a liner wall of zero thickness. We then present a numerical study of the frequency dependence of the coupling impedance of a transverse rectangular slot. Numerical results for a small square hole are presented for frequencies above and below cutoff, and compared with the results of other calculations. [S1063-651X(97)07109-2]

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I. INTRODUCTION

The pumping slots in the liner are the chamber discontinuities, and electromagnetic fields diffracted by them can affect beam stability. This beam-chamber interaction can be described in terms of the coupling impedance. The conventional treatment of the coupling impedance using the static approximation is not sufficient, since for long narrow slots the effect of finite wavelength becomes important [1]. As a first step in obtaining results for a rectangular slot of arbitrary dimensions, in Sec. II we consider a liner with a symmetric annular slot in an inner conductor of negligible thickness (Fig. 1). This analysis serves to illuminate the physics of the problem and to provide a method to obtain accurate numerical results. For low frequencies, the numerical results are checked against analytical results, with which they agree. In Sec. III we treat the same coaxial waveguide as earlier, but this time consider the azimuthally asymmetric problem of a single rectangular slot in the inner conductor (liner), whose thickness is again negligible (Fig. 4). We note that our analysis can be easily extended to any number of slots.

We consider a point charge I_0, traveling along the axis at ultrarelativistic speed. We then calculate the coupling impedance, which turns out to be closely related to the Fourier transform of the wake function, as a function of frequency. The solution is based on the method of field matching at the liner radius, including the discontinuity. We construct a variational form for the impedance, which is stationary with respect to arbitrary small variations of the field about its true value. With such an expression it is possible, by judiciously choosing a trial field, to obtain very accurate results. The variational approach ensures very good accuracy for the impedance, since the error will be proportional to the square of the error in the chosen trial fields. A concomitant advantage is that the variational technique allows us to obtain accurate numerical results with matrices of modest size.

Since the driving current on axis is proportional to $\exp(-jkz)$, the problem is simplified by obtaining results for an even driving current $\cos kz$ and an odd driving current $-j\sin kz$ separately. This separation is needed to construct a variational form for the impedance.

A. Field matching method

The field matching is performed at the radius of the inner conductor (liner) in the opening. We call the region inside the inner conductor $r\leqslant a$ the ''pipe region'' and the region outside the inner conductor $a\leqslant r\leqslant b$ the ''coaxial region.'' The technique consists of expanding fields in both regions into a complete set of functions. At the common interface the fields have to be matched, yielding equations for the expansion coefficients. The resulting integrals contain ratios of the Bessel function and its derivative, which are then expanded into algebraic series of Bessel function zeros, and the resulting integrals are evaluated by means of residue calculus. The

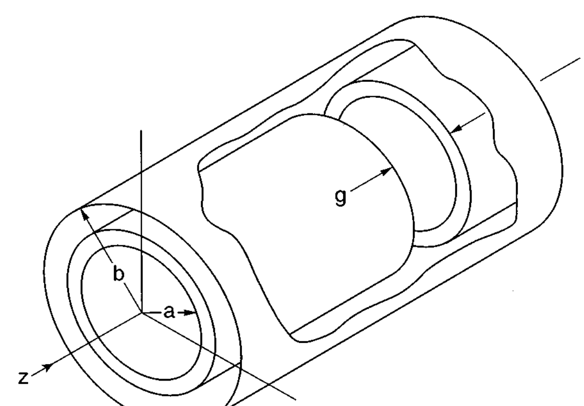

FIG. 1. Schematic diagram of an annular cut in a thin liner.

solution is then obtained by finally truncating and inverting the resulting matrix equations.

B. The longitudinal coupling impedance

The source fields in the frequency domain generated by the driving current

$$J_z(x,y,z;k)=I_0\delta(x)\delta(y)e^{-jkz} \tag{1.1}$$

are the following:

$$E_r^{(s)}(r,z;k)=Z_0H_\theta^{(s)}(r,z;k)=\frac{Z_0I_0}{2\pi r}e^{-jkz}, \tag{1.2}$$

$$E_z^{(s)}(r,z;k)=0, \tag{1.3}$$

where $Z_0=120\pi$ (units Ω), $k=\omega/c$. The definition of the frequency dependent longitudinal coupling impedance of any obstacle can be taken to be

$$Z_\parallel(k)=\frac{-1}{I_0}\int_{-\infty}^{\infty}dze^{jkz}E_z(0,\theta,z;k), \tag{1.4}$$

where $E_z(r,\theta,z;k)$ is the axial electric field in the frequency domain, with frequency dependence $\exp(j\omega t)$, where $\omega=kc$. The field component $E_z(r,\theta,z;k)$ can be written in the pipe region as

$$E_z(r,\theta,z;k)=\sum_n\int_{-\infty}^{\infty}dqe^{-jqz-jn\theta}A_n(q)\frac{J_n(\kappa r)}{J_n(\kappa a)}, \tag{1.5}$$

which is the general solution of the wave equation that is regular at $r=0$. Here κ, defined by $\kappa^2=k^2-q^2$, is the radial propagation constant, and a is the radius of the liner. The contour goes below any poles on the negative real q axis and above any poles on the positive real q axis in order to satisfy the outgoing wave boundary condition for the fields generated by the obstacle.

If we set $r=0$, only the $n=0$ term survives, and Eq. (1.4) becomes

$$Z_\parallel(k)=-\frac{1}{I_0}\int_{-\infty}^{\infty}\frac{dqA_0(q)}{J_0(\kappa a)}\int_{-\infty}^{\infty}dze^{-jz(q-k)}=-\frac{2\pi}{I_0}A_0(k), \tag{1.6}$$

where we used

$$\int_{-\infty}^{\infty}dze^{-jz(q-k)}=2\pi\delta(q-k). \tag{1.7}$$

For an obstacle configuration which does not extend into the pipe ($r<a$), one can perform a Fourier inversion of Eq. (1.5) in z and θ for $r=a$ to obtain

$$A_n(q)=\frac{1}{4\pi^2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dze^{jqz+jn\theta}E_z(a,\theta,z;k). \tag{1.8}$$

Therefore, the longitudinal coupling impedance in Eq. (1.6) can be written as

$$\frac{Z_\parallel(k)}{Z_0}=-\frac{1}{2\pi aZ_0I_0}\int dSE_z(a,\theta,z;k)e^{jkz}, \tag{1.9}$$

where the surface integral is only over the hole, since E_z vanishes on the liner wall. In Sec. II, where we consider the azimuthally symmetric problem, the above expression becomes

$$\frac{Z_\parallel(k)}{Z_0}=-\frac{1}{Z_0I_0}\int dzE_z(a,z;k)e^{jkz}. \tag{1.10}$$

Since the driving current on axis is proportional to $\exp(-jkz)$, the problem is simplified by obtaining $Z_\parallel(k)$ for an even driving current $\cos kz$ and an odd driving current $-j\sin kz$ separately and then taking their sum. We should note that the variational method becomes possible only when the problem is separated into an even and an odd part. In the even problem E_z, H_r, H_θ are even in z, while in the odd problem E_z, H_r, H_θ are odd in z (where $z=0$ is chosen to be the center of the hole). In any case E_z, E_r, H_θ are always even in θ, and H_z, H_r, E_θ are always odd in θ. We use the superscript (e) for the even problem and the superscript (o) for the odd problem.

II. LONGITUDINAL COUPLING IMPEDANCE OF AN ANNULAR CUT IN A COAXIAL LINER

A. General analysis

1. Even part

A schematic diagram of our geometry is shown in Fig. 1. In the pipe region the fields are given by the source fields plus a general solution of the Maxwell equations for the cylindrical waveguide. In the coaxial region we have the general solution of the Maxwell equations for the coaxial waveguide. Due to the symmetry of the problem we have no θ dependence, and therefore need to consider only the azimuthally symmetric TM modes, for which

$$E_z^{(e)}(r,z)=\int dq(\cos qz)A^{(e)}(q)\left[\frac{J_0(\kappa r)}{J_0(\kappa a)},\frac{F_0(\kappa r)}{F_0(\kappa a)}\right]. \tag{2.1}$$

Here we use the notation where the first part in square brackets corresponds to the pipe region $r\leqslant a$, and the second part corresponds to the coaxial region $a\leqslant r\leqslant b$, with the function F_0 being the solution of the Maxwell equations for the coaxial region for the TM modes $[F_0(u)=Y_0(u)J_0(\kappa b)-J_0(u)Y_0(\kappa b)]$. Note that we consider the inside and outside surfaces of the liner both to be at $r=a$, since we neglect the thickness of the liner compared to the slot width. Therefore, the coefficient $A^{(e)}(q)$ is the same for both $r<a$ and $r>a$, since $E_z^{(e)}$ is continuous at $r=a$ within the hole and on both sides of the liner surface, where $E_z^{(e)}=0$. The corresponding azimuthal magnetic field is given by

$$Z_0H_\theta^{(e)}(r,z)=-jk\int dq(\cos qz)A^{(e)}(q)\times\left[\frac{J_0'(\kappa r)}{\kappa J_0(\kappa a)},\frac{F_0'(\kappa r)}{\kappa F_0(\kappa a)}\right]+\left[\frac{Z_0I_0}{2\pi r}(\cos kz),0\right]. \tag{2.2}$$

For the coefficient $A^{(e)}(q)$ we obtain the even part of Eq. (1.8), namely,

$$A^{(e)}(q)=\frac{1}{2\pi}\int dz(\cos qz)E_z^{(e)}(a,z). \tag{2.3}$$

The next step is to match the tangential magnetic field components within the hole. The continuity of $H_\theta^{(e)}$ in the hole gives

$$Z_0I_0\cos kz=jka^2\int dz'E_{z'}^{(e)}(a,z')\int dq(\cos qz)\times(\cos qz')P(q), \tag{2.4}$$

where

$$P(q)=\left[\frac{J_0'(\kappa a)}{\kappa aJ_0(\kappa a)}-\frac{F_0'(\kappa a)}{\kappa aF_0(\kappa a)}\right]. \tag{2.5}$$

We can rewrite Eq. (2.4) in the following form,

$$\int dz'E_{z'}^{(e)}(a,z')K_{11}^{(e)}(z,z')=Z_0I_0(\cos kz), \tag{2.6}$$

where

$$K_{11}^{(e)}(z,z')=K_{11}^{(e)}(z',z)=a\int dq(\cos qz)(\cos qz')k_{11}, \tag{2.7}$$

with

$$k_{11}=jkaP(q). \tag{2.8}$$

We now treat $P(q)$ as a function of κa with $\kappa b=(b/a)\kappa a$ and express this function as a sum over the zeros of the respective denominators. The final result for these expansions is the following:

$$P(q)=-2\sum_{s=1}^{\infty}\frac{1}{q^2a^2-b_s^2}-\left(\frac{1}{\ln(b/a)}\right)\frac{1}{q^2a^2-k^2a^2}+2\sum_{s=1}^{\infty}\frac{\alpha_s}{q^2a^2-c_s^2}, \tag{2.9}$$

where

$$q^2a^2=k^2a^2-\kappa^2a^2,\quad b_s^2=k^2a^2-p_s^2,$$

$$J_0(p_s)=0, \tag{2.10}$$

$$c_s^2=k^2a^2-\sigma_s^2,\quad F_0(\sigma_s)=0, \tag{2.11}$$

and α_s is given by the following expression:

$$\alpha_s=\frac{J_0^2(\sigma_sb/a)}{J_0^2(\sigma_sb/a)-J_0^2(\sigma_s)},\quad s\geqslant 1. \tag{2.12}$$

Here $F_0(\sigma)$ is given by

$$F_0(\sigma)=Y_0(\sigma)J_0(\sigma b/a)-J_0(\sigma)Y_0(\sigma b/a), \tag{2.13}$$

in contrast to our earlier definition of $F_0(u)$ needed to define $F_0'(u)$. The resulting expression for $K_{11}^{(e)}$, in Eq. (2.7), can then be integrated over q by means of the residue theorem.

If we now multiply Eq. (2.6) by $E_z^{(e)}(a,z)$ and integrate both sides over z, then divide by $[\int dzE_z^{(e)}(a,z)(\cos kz)]^2$, we obtain

$$\frac{Z_0I_0}{\int dzE_z^{(e)}(a,z)(\cos kz)}=\left(\int\int dz'dzE_{z'}^{(e)}(a,z')E_z^{(e)}(a,z)K_{11}^{(e)}(z,z')\right)\Big/\left(\int dzE_z^{(e)}(a,z)(\cos kz)\right)^2. \tag{2.14}$$

Using the definition of the impedance for the even part, we can rewrite Eq. (2.14) above as

$$\frac{Z_0}{Z_\parallel^{(e)}}=-\frac{[\int\int dz'dzE_{z'}^{(e)}(a,z')E_z^{(e)}(a,z)K_{11}^{(e)}(z,z')]}{[\int dzE_z^{(e)}(a,z)(\cos kz)]^2}, \tag{2.15}$$

which is easily seen to be a variational form for the impedance, with trial function $E_z^{(e)}(a,z)$.

To proceed from the variational statement of the problem to the solution, we expand the unknown field $E_z^{(e)}(a,z)$ in terms of a complete set of characteristic functions of the hole. For the case of the even problem, we choose

$$E_z^{(e)}(a,z)=\sum_\nu a_\nu\cos\frac{\nu\pi}{g}z, \tag{2.16}$$

with ν being even so that $\partial E_z/\partial z=0$ at $z=\pm g/2$. In Eq. (2.16), g is the width of the slot in the longitudinal direction z.

After evaluation of the integrals in Eq. (2.15), the solution for the even part of the impedance is obtained by finally truncating and inverting the resulting matrix equations.

2. Odd part

We now consider the portion of the problem when $E_z^{(o)}$ is odd in z, using the same notation as for the even part.

$$E_z^{(o)}(r,z)=-j\int dq(\sin qz)A^{(o)}(q)\left[\frac{J_0(\kappa r)}{J_0(\kappa a)},\frac{F_0(\kappa r)}{F_0(\kappa a)}\right]. \tag{2.17}$$

$$Z_0H_\theta^{(o)}(r,z)=-k\int dq(\sin qz)A^{(o)}(q)\times\left[\frac{J_0'(\kappa r)}{\kappa J_0(\kappa a)},\frac{F_0'(\kappa r)}{\kappa F_0(\kappa a)}\right]-\left[j\frac{Z_0I_0}{2\pi r}(\sin kz),0\right]. \tag{2.18}$$

For the coefficient $A^{(o)}(q)$ we obtain the odd part of Eq. (1.8), namely,

$$A^{(o)}(q)=j\frac{1}{2\pi}\int dz(\sin qz)E_z^{(o)}(a,z). \tag{2.19}$$

The next step is to match the tangential magnetic field components within the hole. The continuity of $H_\theta^{(o)}$ in the hole gives

$$-jZ_0I_0(\sin kz)=jka^2\int dz'E_{z'}^{(o)}(a,z')\int dq(\sin qz)\times(\sin qz')P(q). \tag{2.20}$$

We can rewrite Eq. (2.20) in the following form

$$\int dz'E_{z'}^{(o)}(a,z')K_{11}^{(o)}(z,z')=-jZ_0I_0(\sin kz), \tag{2.21}$$

where

$$K_{11}^{(o)}(z,z')=K_{11}^{(o)}(z',z)=a\int dq(\sin qz)(\sin qz')k_{11}, \tag{2.22}$$

with k_{11} given by Eq. (2.8). If we now multiply Eq. (2.21) by $E_z^{(o)}(a,z)$ and integrate both sides over z, then divide by $[\int dzE_z^{(o)}(a,z)(\sin kz)]^2$, we obtain

$$\frac{-jZ_0I_0}{\int dzE_z^{(o)}(a,z)(\sin kz)}=\left(\int\int dz'dzE_{z'}^{(o)}(a,z')E_z^{(o)}(a,z)K_{11}^{(o)}(z,z')\right)\Big/\left(\int dzE_z^{(o)}(a,z)(\sin kz)\right)^2. \tag{2.23}$$

Using the definition of the impedance for the odd part, we can rewrite Eq. (2.23) above as

$$\frac{Z_0}{Z_\parallel^{(o)}}=-\frac{[\int\int dz'dzE_{z'}^{(o)}(a,z')E_z^{(o)}(a,z)K_{11}^{(o)}(z,z')]}{[\int dzE_z^{(o)}(a,z)(\sin kz)]^2}, \tag{2.24}$$

which is a variational form for the odd part of the impedance.

To proceed from the variational statement of the problem to the solution, we expand the unknown field $E_z^{(o)}(a,z)$ in terms of a complete set of characteristic functions of the hole. For the case of odd problem, we choose

$$E_z^{(o)}(a,z)=\sum_\nu a_\nu\sin\left(\frac{\nu\pi}{g}z\right), \tag{2.25}$$

with ν being odd so that $\partial E_z/\partial z=0$ again at $z=\pm g/2$. After evaluation of the integrals in Eq. (2.24), the solution for the odd part of the impedance is obtained by finally truncating and inverting the resulting matrix equations.

B. Analytic derivation for low frequencies

The even part of the impedance can be related to the magnetic susceptibility of the hole. The odd part of the impedance can be related to the electric polarizability of the hole. For the transverse narrow slot in the Bethe approximation for a small hole, the magnetic susceptibility is proportional to l^3, where l is the azimuthal length of the slot, and the electric polarizability is proportional to lw^2, where w is the width of the slot (see, for example, [1,2]). Therefore, one would expect the odd part to be negligible as long as $l\gg w$. Numerical study for slots with azimuthal length larger than the radius of the curvature of the pipe in Sec. III showed that the argument discussed above holds even when $l>a$ (with a being the radius of the inner conductor), where the small hole approximation already fails.

In our particular case where the slot is a narrow annular cut, for low frequencies the leading nonvanishing term of the odd part is a factor of $(g/a)^2$ less than the same term in the even part. This can be seen by performing an analysis for the odd part similar to the analysis given below. To present the analytic result for a narrow annular cut it is therefore sufficient to consider only the even part of the impedance.

From Eq. (2.15), the variational form for the even part of impedance is the following:

$$\frac{Z_\parallel^{(e)}}{Z_0}=\frac{j}{ka^2}\left(\int dzE_z^{(e)}(a,z)(\cos kz)\right)^2\Big/\left(\int\int dz'dzE_{z'}^{(e)}(a,z')E_z^{(e)}(a,z)\times\left[\int dq(\cos qz)(\cos qz')P(q)\right]\right), \tag{2.26}$$

where $P(q)$ is given by Eq. (2.5). For small ka we write the quantities in Eqs. (2.10) and (2.11) as

$$b_s^2=-p_s^2,\quad b_s=-jp_s,\quad c_s^2=-\sigma_s^2,$$

$$c_s=-j\sigma_s,\quad c_0^2=k^2a^2. \tag{2.27}$$

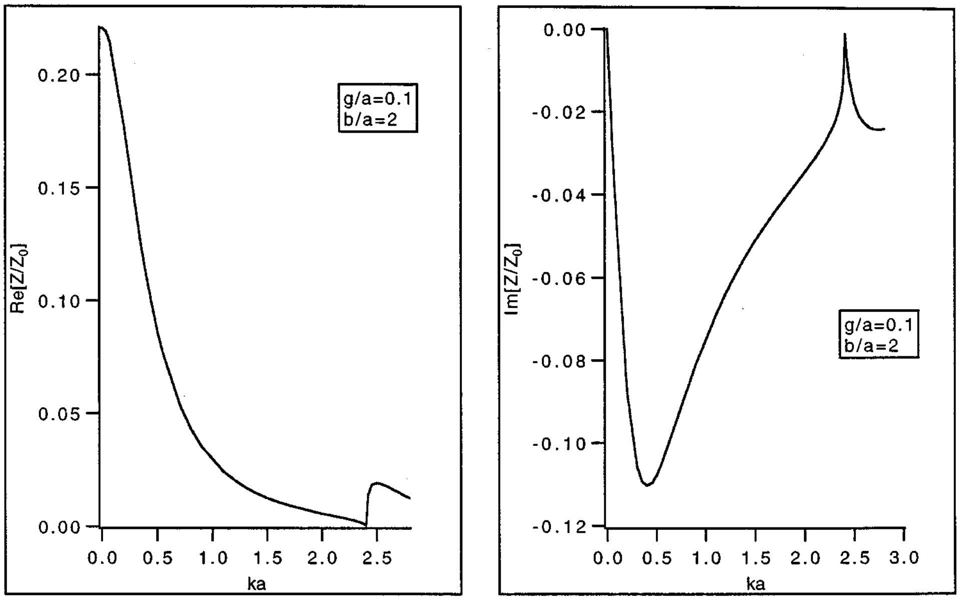

FIG. 2. Real and imaginary parts of the coupling impedance of an annular cut in a coaxial liner.

By integrating over q in the square brackets of Eq. (2.26), we obtain

$$-\sum_{s=1}\frac{\pi}{p_s a}[\exp(-\lambda p_s/a)+\exp(-\tilde{\lambda}p_s/a)]$$
$$-\frac{j\alpha_0\pi}{ka^2}[\exp(-j\lambda k)+\exp(-j\tilde{\lambda}k)]$$
$$+\sum_{s=1}\frac{\pi\alpha_s}{\sigma_s a}[\exp(-\lambda\sigma_s/a)+\exp(-\tilde{\lambda}\sigma_s/a)], \tag{2.28}$$

where α_s is given by Eq. (2.12) and $\alpha_0=-1/[2\ln(b/a)]$. Here we use notation

$$\lambda=|z+z'|,\quad \tilde{\lambda}=|z-z'|. \tag{2.29}$$

For small λ, $\tilde{\lambda}$, $z\pm z'$ we can replace the sums in Eq. (2.28) by integrals. By evaluating the integrals, Eq. (2.28) becomes

$$-\frac{1}{a}\left[\ln\left(\frac{a^2}{\lambda\tilde{\lambda}}\right)+2C_1\right]-\frac{1}{a}\left[\ln\left(\frac{a^2}{\lambda\tilde{\lambda}}\right)+2C_2\right]$$
$$-\frac{-j\pi\alpha_0}{ka^2}(2-jk|z-z'|-jk|z+z'|), \tag{2.30}$$

where constants of integration C_1 and C_2 can be found by simple numerical calculation. The resulting expression for the impedance becomes

$$\frac{Z_\parallel^{(e)}}{Z_0}=\frac{[\int dz E_z^{(e)}(a,z)]^2}{\int\int dz'dz E_{z'}^{(e)}(a,z')E_z^{(e)}(a,z)M(z,z')}, \tag{2.31}$$

where

$$M(z,z')=\left[-2\pi\alpha_0+2jka(C_1+C_2)+2jka\ \ln\frac{a^2}{|z^2+(z')^2|}\right.$$
$$\left.+j\pi\alpha_0 k(|z+z'|+|z-z'|)\right]. \tag{2.32}$$

In Eq. (2.31) the integrals over z and z' go from $-g/2$ to $g/2$, with g being the width of the cut in the longitudinal direction. After evaluating the integrals in Eq. (2.31), using the static approximation for $E_{z'}$, E_z, we obtain

$$\frac{Z_\parallel}{Z_0}=\frac{\ln(b/a)}{\pi}\left[1+j\ \frac{4}{\pi^2}kag/a\right.$$
$$\left.-2jka\frac{\ln(b/a)}{\pi}[C_1+C_2+2\ln(4a/g)]\right]. \tag{2.33}$$

In this result we suppressed the superscript (e), since the odd part of the impedance is negligible. Numerical study shows that C_1+C_2 can be replaced by $\ln(b/a-1)-1.78$ for the range of b/a from 1 to 3.

C. Numerical results

Formulas for numerical computation of the impedance of an annular slot in the inner conductor of coaxial structure are obtained. The even part of the impedance is calculated using Eq. (2.15) and the odd part is calculated using Eq. (2.24).

TABLE I. $\mathrm{Im}(Z_\parallel/Z_0)$ for $b/a=2$ at frequency $ka=0.03$, $C_1=-0.667$, $C_2=-1.117$.

g/a	Analytic approximation	Numerical result
0.01	0.02976	0.02907
0.03	0.02332	0.02308
0.05	0.02025	0.02022
0.07	0.01823	0.01830

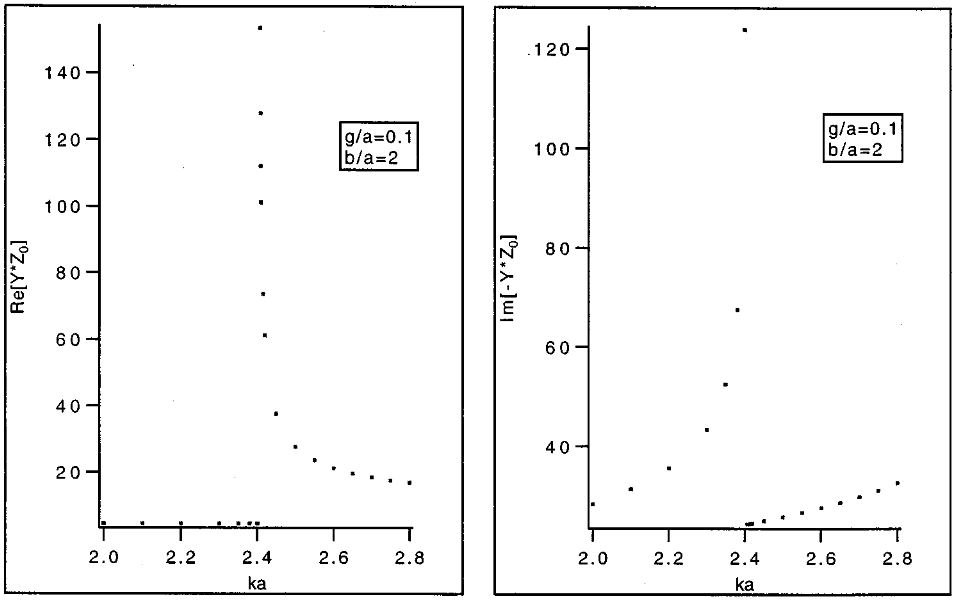

FIG. 3. Real and imaginary parts of the admittance of an annular cut in a coaxial liner.

Finally, one sums together the even and the odd parts to obtain the coupling impedance. It turns out that for the case of the annular cut, which we consider in Sec. II, the odd part is negligible. The same is true even for the slot with finite azimuthal length as long as the azimuthal length is much larger than the width of the slot (see Sec. III). The general behavior of the real and imaginary parts with respect to frequency is presented in Fig. 2.

D. Discussion

The interesting feature of the annular cut is that in the limit of zero frequency the real part becomes finite, and the imaginary part becomes capacitive. For low frequencies we obtain the leading terms for the real and imaginary parts analytically, according to Eq. (2.33). The agreement between the analytic and numerical results is very good. As an example, results for $b/a=2$ are presented in Table I. For other values of b/a, the analytic and numerical results are also in good agreement.

The real part of the impedance in the limit of zero frequency becomes finite, and is equal to $\ln(b/a)/\pi$, which agrees with the result obtained by Palumbo [3]. Its physical origin is the energy radiated in the TEM mode in the coaxial region. With a small hole in the wall, ''image'' current lines change their profile very little as they avoid the hole, generating almost no radiation. When the pipe is truncated (annular cut), the current lines are simply interrupted, leading to the significant radiation in the coaxial region.

For low frequencies, the coupling impedance of the narrow annular cut can be easily presented in terms of an equivalent circuit. Specifically, for $g \ll a$, we can write for the admittance

$$Y=R^{-1}+j\omega C, \tag{2.34}$$

where R is given by $Z_0\ln(b/a)/\pi$, and C is given by $2a\epsilon_0[C_1+C_2+2\ln(4a/g)]$, corresponding to the parallel combination of the resistance R and the capacitance C. In Fig. 3 we present the real and imaginary parts of the admittance Y as a function of ka. As one can see, the real part of the admittance is purely $1/R$ until $ka=2.405$, the cutoff of the TM_{01} mode. At this cutoff the singularity corresponds to the fact that power starts to dissipate not just in the coaxial region, but also in the pipe region.

III. LONGITUDINAL COUPLING IMPEDANCE OF A RECTANGULAR SLOT

A. The even part of impedance

1. General analysis

A schematic diagram of our geometry with the rectangular slot is shown in Fig. 4. In the pipe region the fields are given by the source fields plus a general solution of the Maxwell equations for the cylindrical waveguide. In the coaxial

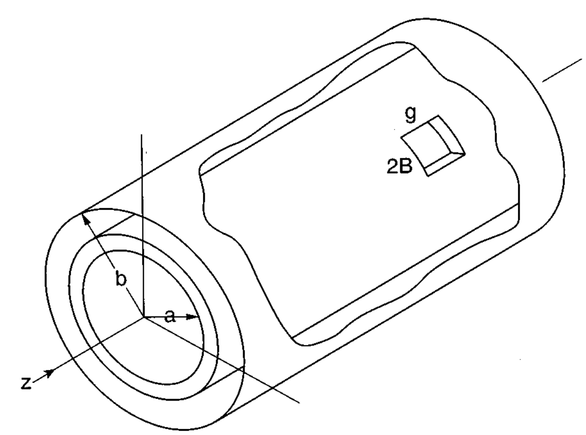

FIG. 4. Schematic diagram of a rectangular slot in a thin liner.

region we have a general solution of the Maxwell equations for the coaxial waveguide. Due to the asymmetry of the problem we now have θ dependence, and therefore need to consider both TM and TE modes. For the TM portion of the modes we have

$$E_z^{(e)}(r,\theta,z)=\int dq(\cos qz)\phi^{(e)}(r,\theta), \tag{3.1}$$

where

$$\phi^{(e)}(r,\theta)=\sum_n (\cos n\theta)A_n^{(e)}(q)\left[\frac{J_n(\kappa r)}{J_n(\kappa a)},\ \frac{F_n(\kappa r)}{F_n(\kappa a)}\right]. \tag{3.2}$$

Here we use the notation where the first part in square brackets corresponds to the pipe region $r\leqslant a$, and the second part corresponds to the coaxial region $a\leqslant r\leqslant b$, with the function F_n being the solution of the Maxwell equations for the coaxial region for the TM modes $[F_n(u)=Y_n(u)J_n(\kappa b)-J_n(u)Y_n(\kappa b)]$. Note that we consider the inside and outside surfaces of the liner both to be at $r=a$, since we neglect the thickness of the liner compared to the wavelength and to the dimensions of the rectangular slot. Therefore, the coefficient $A_n^{(e)}(q)$ is the same for both $r<a$ and $r>a$, since $E_z^{(e)}$ is continuous at $r=a$ within the hole and on both sides of the liner surface, where $E_z^{(e)}=0$. The other corresponding TM field components can be obtained as

$$E_r^{(e)}(r,\theta,z)=-\int dq\frac{q(\sin qz)}{\kappa^2}\frac{\partial\phi^{(e)}(r,\theta)}{\partial r}-j\frac{Z_0I_0}{2\pi r}[(\sin kz),0], \tag{3.3}$$

$$E_\theta^{(e)}(r,\theta,z)=-\int dq\frac{q(\sin qz)}{\kappa^2 r}\frac{\partial\phi^{(e)}(r,\theta)}{\partial\theta}, \tag{3.4}$$

$$Z_0H_\theta^{(e)}(r,\theta,z)=-jk\int \frac{dq(\cos qz)}{\kappa^2}\frac{\partial\phi^{(e)}(r,\theta)}{\partial r}+\frac{Z_0I_0}{2\pi r}[(\cos kz),0], \tag{3.5}$$

$$Z_0H_r^{(e)}(r,\theta,z)=jk\int \frac{dq(\cos qz)}{\kappa^2 r}\frac{\partial\phi^{(e)}(r,\theta)}{\partial\theta}. \tag{3.6}$$

For the TE portion of the modes we similarly have

$$Z_0H_z^{(e)}(r,\theta,z)=\int dq(\sin qz)\psi^{(e)}(r,\theta), \tag{3.7}$$

where

$$\psi^{(e)}(r,\theta)=-\sum_n (\sin n\theta)B_n^{(e)}\left[\frac{J_n(\kappa r)}{J_n'(\kappa a)},\frac{G_n(\kappa r)}{G_n'(\kappa a)}\right], \tag{3.8}$$

with the function $G_n(\kappa r)$ being the solution of the Maxwell equations for the coaxial region for the TE modes $[G_n(u)=Y_n(u)J_n'(\kappa b)-J_n(u)Y_n'(\kappa b)]$. The other TE field components can be obtained as

$$Z_0H_r^{(e)}(r,\theta,z)=\int \frac{dqq(\cos qz)}{\kappa^2}\frac{\partial\psi^{(e)}(r,\theta)}{\partial r}, \tag{3.9}$$

$$Z_0H_\theta^{(e)}(r,\theta,z)=\int \frac{dqq(\cos qz)}{\kappa^2 r}\frac{\partial\psi^{(e)}(r,\theta)}{\partial\theta}, \tag{3.10}$$

$$E_\theta^{(e)}(r,\theta,z)=jk\int \frac{dq(\sin qz)}{\kappa^2}\frac{\partial\psi^{(e)}(r,\theta)}{\partial r}, \tag{3.11}$$

$$E_r^{(e)}(r,\theta,z)=-jk\int \frac{dq(\sin qz)}{\kappa^2 r}\frac{\partial\psi^{(e)}(r,\theta)}{\partial\theta}. \tag{3.12}$$

For the general solution, we include both the TM and TE portions of the modes, and write explicitly

$$Z_0H_\theta^{(e)}(r,\theta,z)=-jk\sum_n\left(\int dq(\cos qz)(\cos n\theta)A_n^{(e)}(q)\left[\frac{J_n'(\kappa r)}{\kappa J_n(\kappa a)},\ \frac{F_n'(\kappa r)}{\kappa F_n(\kappa a)}\right]\right)+\frac{Z_0I_0}{2\pi r}[(\cos kz),0]-\sum_n\left(\int dq\frac{qn}{r\kappa^2}(\cos qz)(\cos n\theta)B_n^{(e)}(q)\left[\frac{J_n(\kappa r)}{J_n'(\kappa a)},\frac{G_n(\kappa r)}{G_n'(\kappa a)}\right]\right), \tag{3.13}$$

$$E_\theta^{(e)}(r,\theta,z)=\sum_n\left(\int dq\frac{qn}{r\kappa^2}(\sin qz)(\sin n\theta)A_n^{(e)}(q)\left[\frac{J_n(\kappa r)}{J_n(\kappa a)},\ \frac{F_n(\kappa r)}{F_n(\kappa a)}\right]\right)-jk\sum_n\left(\int dq(\sin qz)(\sin n\theta)B_n^{(e)}(q)\left[\frac{J_n'(\kappa r)}{\kappa J_n'(\kappa a)},\ \frac{G_n'(\kappa r)}{\kappa G_n'(\kappa a)}\right]\right). \tag{3.14}$$

We can express $A_n^{(e)}(q)$ in terms of the $E_z^{(e)}(a,\theta,z)$ by using the even part of Eq. (1.8) to obtain

$$A_n^{(e)}(q)=\frac{1}{4\pi^2 a}\int dS(\cos qz)(\cos n\theta)E_z^{(e)}(a,\theta,z). \tag{3.15}$$

Similarly we express $B_n^{(e)}(q)$ in terms of $E_z^{(e)}(a,\theta,z)$ and $E_\theta^{(e)}(a,\theta,z)$ by inverting Eq. (3.14) at $r=a$. Specifically

$$\frac{qn}{a\kappa^2}A_n^{(e)}(q)-jk\frac{B_n^{(e)}(q)}{\kappa}=\frac{1}{4\pi^2 a}\int dS(\sin qz)(\sin n\theta)\times E_\theta^{(e)}(a,\theta,z). \tag{3.16}$$

Therefore we find for the coefficient $B_n^{(e)}(q)$

$$B_n^{(e)}(q)=j\frac{\kappa}{k}\frac{1}{4\pi^2 a}\int dS\left[(\sin qz)(\sin n\theta)E_\theta^{(e)}(a,\theta,z)-\frac{qn}{a\kappa^2}(\cos qz)(\cos n\theta)E_z^{(e)}(a,\theta,z)\right]. \tag{3.17}$$

The next step is to match the tangential magnetic field components within the hole. The continuity of $H_z^{(e)}$ and $H_\theta^{(e)}$ in the hole at $r=a$ leads to

$$\int dS' E_{\theta'}^{(e)}(a,\theta',z')K_{22}^{(e)}+\int dS' E_{z'}^{(e)}(a,\theta',z')K_{21}^{(e)}=0, \tag{3.18}$$

$$\int dS' E_{z'}^{(e)}(a,\theta',z')K_{11}^{(e)}+\int dS' E_{\theta'}^{(e)}(a,\theta',z')K_{12}^{(e)}=aZ_0 I_0 2\pi(\cos kz), \tag{3.19}$$

where

$$K_{11}^{(e)}=a\sum_n\int dq(\cos qz)(\cos qz')(\cos n\theta)(\cos n\theta')k_{11}, \tag{3.20}$$

$$K_{12}^{(e)}=K_{21}^{(e)}=a\sum_n\int dq(\cos qz)(\sin qz')(\sin n\theta')(\cos n\theta)k_{12}, \tag{3.21}$$

$$K_{22}^{(e)}=a\sum_n\int dq(\sin qz)(\sin qz')(\sin n\theta)(\sin n\theta')k_{22}, \tag{3.22}$$

with

$$k_{11}=jkaP_n(q)-j\frac{q^2n^2}{\kappa^2 ka}Q_n(q), \tag{3.23}$$

$$k_{12}=k_{21}=j\frac{qn}{k}Q_n(q), \tag{3.24}$$

$$k_{22}=-j\frac{\kappa^2 a}{k}Q_n(q). \tag{3.25}$$

Here the functions $P_n(q)$ and $Q_n(q)$ are given by the following expressions

$$P_n(q)=\left[\frac{J_n'(\kappa a)}{\kappa aJ_n(\kappa a)}-\frac{F_n'(\kappa a)}{\kappa aF_n(\kappa a)}\right], \tag{3.26}$$

$$Q_n(q)=\left[\frac{J_n(\kappa a)}{\kappa aJ_n'(\kappa a)}-\frac{G_n(\kappa a)}{\kappa aG_n'(\kappa a)}\right], \tag{3.27}$$

where $\kappa^2=k^2-q^2$.

We now treat $P_n(q)$ and $Q_n(q)$ as functions of κa with $\kappa b=(b/a)\kappa a$ and express these functions as a sum over the zeros of the respective denominators. The detailed expansion of the functions $P_n(q)$ and $Q_n(q)$ in terms of algebraic series is given in Appendix B. The resulting expressions for $K_{11}^{(e)}$, $K_{12}^{(e)}$, $K_{21}^{(e)}$, and $K_{22}^{(e)}$, in Eqs. (3.20)–(3.22) can then be integrated over q by means of the residue theorem.

2. The variational form

From Eq. (1.9) the even part of the impedance is

$$\frac{Z_\parallel^{(e)}}{Z_0}=\frac{-1}{2\pi aZ_0I_0}\int dSE_z^{(e)}(a,\theta,z)(\cos kz). \tag{3.28}$$

Using Eqs. (3.18) and (3.19) we can form

$$\frac{Z_0}{Z_\parallel^{(e)}}=-\left(\int\int dS'dS[2E_{\theta'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{12}^{(e)}+E_{z'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{11}^{(e)}+E_{\theta'}^{(e)}(a,\theta',z')E_\theta^{(e)}(a,\theta,z)K_{22}^{(e)}]\right)\Bigg/\left(\int dSE_z^{(e)}(a,\theta,z)(\cos kz)^2\right), \tag{3.29}$$

a form independent of the normalization of the fields. If we ask that the numerator of Eq. (3.29) be a minimum with respect to the variations of $E_z^{(e)}$ and $E_\theta^{(e)}$, subject to the constraint $\int dSE_z^{(e)}(a,\theta,z)\cos kz=1$, we can use the method of Lagrange multipliers to form

$$J=\int\int dS'dS[2E_{\theta'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{12}^{(e)} +E_{z'}^{(e)}(a,\theta',z')E_z^{(e)}(a,\theta,z)K_{11}^{(e)}+E_{\theta'}^{(e)}(a,\theta',z')E_\theta^{(e)} \times(a,\theta,z)K_{22}^{(e)}]-2\lambda\int dSE_z^{(e)}(a,\theta,z)(\cos kz) =\text{minimum}, \tag{3.30}$$

and require that $\delta J=0$ for variations in $\delta E_z^{(e)}$ and $\delta E_\theta^{(e)}$. By doing so we reproduce Eqs. (3.18) and (3.19) and confirm that Eq. (3.29) is a variational form for the impedance.

We now proceed from the variational statement of the problem to the solution. The first step is the expansion of the unknown fields $E_z^{(e)}$ and $E_\theta^{(e)}$ in terms of a complete set of functions characteristic of the hole. Using the expansions

$$E_z^{(e)}(a,\theta,z)=\sum a_\sigma u_\sigma(\theta,z) \tag{3.31}$$

and

$$E_\theta^{(e)}(a,\theta,z)=\sum b_\sigma v_\sigma(\theta,z), \tag{3.32}$$

where u_σ and v_σ are each an orthonormal set of functions in the hole, we then have

$$J=\sum a_\sigma a_{\sigma'}L_{\sigma\sigma'}^{(e)}+2a_\sigma b_{\sigma'}M_{\sigma\sigma'}^{(e)}+b_\sigma b_{\sigma'}N_{\sigma\sigma'}^{(e)} -2\lambda\sum a_\sigma P_\sigma^{(e)}, \tag{3.33}$$

with $L^{(e)}$, $M^{(e)}$, $N^{(e)}$, and $P^{(e)}$ defined in Appendix A. The vanishing of the partial derivative with respect to $\boldsymbol{a}$ gives

$$\sum a_{\sigma'}L_{\sigma\sigma'}^{(e)}+b_{\sigma'}M_{\sigma\sigma'}^{(e)}=\lambda P_\sigma^{(e)}. \tag{3.34}$$

The vanishing of the partial derivative with respect to $\boldsymbol{b}$ gives

$$\sum a_{\sigma'}M_{\sigma\sigma'}^{(e)}+b_{\sigma'}N_{\sigma\sigma'}^{(e)}=0. \tag{3.35}$$

We solve these two matrix equations to obtain

$$\boldsymbol{a}=\lambda H^{(e)^{-1}}P^{(e)}, \tag{3.36}$$

where the matrix $H^{(e)}$ in Eq. (3.36) is defined as

$$H^{(e)}=L^{(e)}-M^{(e)}N^{(e)^{-1}}\widetilde{M}^{(e)}. \tag{3.37}$$

Therefore we can write

$$\frac{Z_0}{Z_\parallel^{(e)}}=\frac{-\lambda}{\lambda P^{(e)}H^{(e)^{-1}}P^{(e)}}=-\frac{1}{P^{(e)}H^{(e)^{-1}}P^{(e)}}, \tag{3.38}$$

independent of the normalization parameter λ. After evaluation of the integrals (see Appendix A), we obtain the final form for the even part of the impedance

$$\frac{Z_\parallel^{(e)}}{Z_0}=\sum_{\nu\nu'\mu\mu'}\left[-\frac{64B^2k^2a^2\sin^2[(ka/2)g/a]}{\mu\mu'\pi^2}\frac{\sin(\mu\pi/2)\sin(\mu'\pi/2)\cos(\nu'\pi/2)\cos(\nu\pi/2)}{[(\nu\pi/g/a)^2-k^2a^2][(\nu'\pi/g/a)^2-k^2a^2]}(H^{(e)})^{-1}_{\nu\nu'\mu\mu'}\right], \tag{3.39}$$

with μ,μ' being odd, and ν,ν' being even. The final forms of the quantities $L^{(e)}$, $M^{(e)}$, and $N^{(e)}$, needed to evaluate the matrix $H^{(e)}$ in Eq. (3.39), are given in Appendix A.

B. The odd part of impedance

1. General analysis

We now consider the portion of the problem where $E_z^{(o)}$ is odd in z. We perform expansion for the fields similar to those for the even portion of the problem. In the expressions for the fields in Eqs. (3.1)–(3.14) we replace coskz by $-j$(sinkz), sinkz by j(coskz), cosqz by $-j$(sinqz), and sinqz by j(cosqz) as governed by the form of Eq. (1.5). In the expressions for the coefficients in Eqs. (3.15)–(3.17) we replace cosqz' by j(sinqz') and sinqz' by $-j$(cosqz') as governed by the form of Eq. (1.8).

The continuity of $H_z^{(o)}$ and $H_\theta^{(o)}$ in the hole at $r=a$ leads to the following integral equations

$$\int dS'E_{\theta'}^{(o)}(a,\theta',z')K_{22}^{(o)}+\int dS'E_{z'}^{(o)}(a,\theta',z')K_{21}^{(o)}=0, \tag{3.40}$$

$$\int dS'E_{z'}^{(o)}(a,\theta',z')K_{11}^{(o)}+\int dS'E_{\theta'}^{(o)}(a,\theta',z')K_{12}^{(o)} =aZ_0I_0 2\pi(-j\sin kz), \tag{3.41}$$

where

$$K_{11}^{(o)}=a\sum_n\int dq(\sin qz)(\sin qz')(\cos n\theta)(\cos n\theta')k_{11}, \tag{3.42}$$

$$K_{12}^{(o)}=K_{21}^{(o)} =-a\sum_n\int dq(\sin qz)(\cos qz')(\cos n\theta)(\sin n\theta')k_{12}, \tag{3.43}$$

$$K_{22}^{(o)}=a\sum_n \int dq(\cos qz)(\cos qz')(\sin n\theta)(\sin n\theta')k_{22}, \tag{3.44}$$

with k_{ij} given by Eqs. (3.23)–(3.25). In Eqs. (3.20)–(3.22) we use both sets of replacements outlined at the start of this section to obtain Eqs. (3.40)–(3.44).

2. The variational form for the odd part

From Eq. (1.9) the odd part of the impedance is

$$\frac{Z_\parallel^{(o)}}{Z_0}=\frac{-j}{2\pi a Z_0 I_0}\int dSE_z^{(o)}(a,\theta,z)(\sin kz). \tag{3.45}$$

Using Eqs. (3.40) and Eq. (3.41) we can form

$$\frac{Z_0}{Z_\parallel^{(o)}}=-\left(\int\int dS'dS[2E_{\theta'}^{(o)}(a,\theta',z')E_z^{(o)}(a,\theta,z)K_{12}^{(o)}+E_{z'}^{(o)}(a,\theta',z')E_z^{(o)}(a,\theta,z)K_{11}^{(o)} +E_{\theta'}^{(o)}(a,\theta',z')E_\theta^{(o)}(a,\theta,z)K_{22}^{(o)}]\right)\Big/\left(\int dSE_z^{(o)}(a,\theta,z)(\sin kz)^2\right). \tag{3.46}$$

If we require that the numerator of Eq. (3.46) be a minimum with respect to the variations of $E_z^{(o)}$ and $E_\theta^{(o)}$, subject to the constraint $\int dSE_z^{(o)}(a,\theta,z)(\sin kz)=1$, as we did for the even part, we find that Eq. (3.46) is the variational form for the odd part of the impedance. After expanding $E_z^{(o)}(a,\theta,z)$ and $E_\theta^{(o)}(a,\theta,z)$ in terms of a complete set of functions characteristic of the hole, we ultimately obtain for the odd part of the impedance

$$\frac{Z_0}{Z_\parallel^{(o)}}=-\frac{1}{P^{(o)}H^{(o)^{-1}}P^{(o)}}, \tag{3.47}$$

where the matrix $H^{(o)}$ in Eq. (3.47) is defined as

$$H^{(o)}=L^{(o)}-M^{(o)}N^{(o)^{-1}}\widetilde{M}^{(o)}. \tag{3.48}$$

After evaluation of the integrals (see Appendix A), we obtain the final form for calculation of the odd part of the impedance

$$\frac{Z_\parallel^{(o)}}{Z_0}=\sum_{\nu\nu'\mu\mu'}\Bigg[\frac{-64B^2k^2a^2\cos^2[(ka/2)g/a]}{\mu\mu'\pi^2}\times\frac{\sin(\mu\pi/2)\sin(\mu'\pi/2)\sin(\nu\pi/2)\sin(\nu'\pi/2)}{[(\nu\pi/(g/a))^2-k^2a^2][(\nu'\pi/(g/a))^2-k^2a^2]}\times(H^{(o)})^{-1}_{\nu\nu'\mu\mu'}\Bigg], \tag{3.49}$$

with μ,μ',ν,ν' being odd. The final expressions for the quantities $L^{(o)}$, $M^{(o)}$, and $N^{(o)}$, needed to evaluate the matrix $H^{(o)}$ in Eq. (3.49), are given in Appendix A.

TABLE II. Im(Z) (units mΩ) for a square hole with 4 mm edge length at frequency 1 GHz, $a=16$ mm.

ka	$b/a=1.2$	$b/a=1.3125$	$b/a=1.5$
0.3351	6.46	6.57	6.60

C. Numerical results and discussions

Formulas for direct numerical computation of the impedance of a rectangular slot have been obtained. The even part of the impedance is calculated using Eq. (3.39). The odd part of the impedance is calculated using Eq. (3.49). Finally, the sum of the even and odd parts yields the result for the coupling impedance.

To obtain suitable numerical accuracy we need a 5×5 matrix for the summation over ν,ν', and a 3×3 matrix for the summation over μ,μ'. These matrix sizes are used to obtain the convergent results in both ν,ν' and μ,μ' in Tables II and III. We estimate the accuracy of the numbers listed to be around 3%. To present the frequency behavior of the impedance in Figs. 5–7 a 5×5 matrix is used for the summation over ν,ν', but only $\mu=\mu'=1$ is taken for convenience. From a few test cases with terms up to $\mu=\mu'=5$, we estimate that the results for only $\mu=\mu'=1$ are approximately 6% low.

1. Transverse rectangular slots

We can use Eqs. (3.39) and (3.49) to study the coupling impedance of a rectangular slot of specific geometry. As an example, and to test our formulas, we present here a numerical study of the impedance of rectangular slots of different azimuthal length. In order to compare our results with those presented by Filtz and Scholz [4] we also choose the parameters of the LHC design.

In Figs. 5 and 6, the frequency dependence of both the real and imaginary parts of the coupling impedance of the transverse rectangular slot is presented, with an angular length 180 and 350 degrees, respectively. As expected, the behavior with respect to frequency strongly differs from the one of slots with the small angular length, even for relatively

TABLE III. Re(Z) (units $\mu\Omega$) for a square hole with 4 mm edge length at frequency 1 GHz, $a=16$ mm.

Re(Z)	$b/a=1.2$	$b/a=1.3125$	$b/a=1.5$
Our result	7.54	5.21	3.53
Scholz's result [6]	≈7.7	≈5.3	≈3.8
Analytic result, using [9]	1.64	1.1	0.74

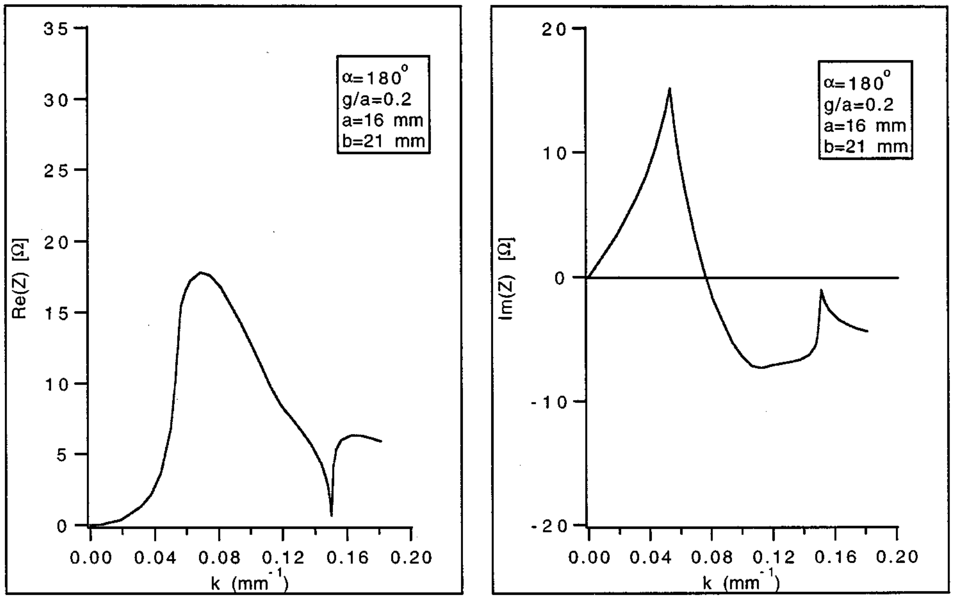

FIG. 5. Real and imaginary parts of the coupling impedance of transverse rectangular slot with the azimuthal length $\alpha \equiv 2B$ equal to 180°.

low frequencies. When we increase the azimuthal length of the slot, the behavior of the impedance becomes similar to the limit when the azimuthal length is equal to 2π (which corresponds to the annular cut in the inner pipe considered in Sec. II), except for very low frequencies. In the limit of the annular cut, for low frequencies, the real part becomes finite, and the imaginary part becomes capacitive. The derivation and detailed explanation of this fact was given in Sec. II.

The general behavior of the impedance agrees with the results of Filtz and Scholz, even though there is some shift between our results and theirs. They used a similar field matching technique, but their calculations of the impedance were performed without using a variational form, presumably requiring large matrices and considerable CPU time. Our approach, which uses a variational form, requires only modest size matrices. The simplicity of the calculation allowed us to perform the present numerical study using MATHEMATICA [5].

2. Small rectangular hole

In order to check our results with the theory developed for small holes and with the calculation of Scholz [6], a numerical study is performed for a liner radius $a=16$ mm for a small square hole with the edge length equal to 4 mm, the parameters used by Scholz. The frequency dependence of both the real and imaginary parts of the coupling impedance is presented in Fig. 7. Similar plots were obtained by Scholz [6], but the peaks he obtains for the cutoffs of the modes in the coaxial region are in error [7].

a. Imaginary part. In the well known Bethe small hole approximation the imaginary part of the impedance below cutoff is given by

$$Z(\omega)=jZ_0\frac{\omega}{c_0}\frac{\alpha_m+\alpha_e}{4\pi^2a^2}, \tag{3.50}$$

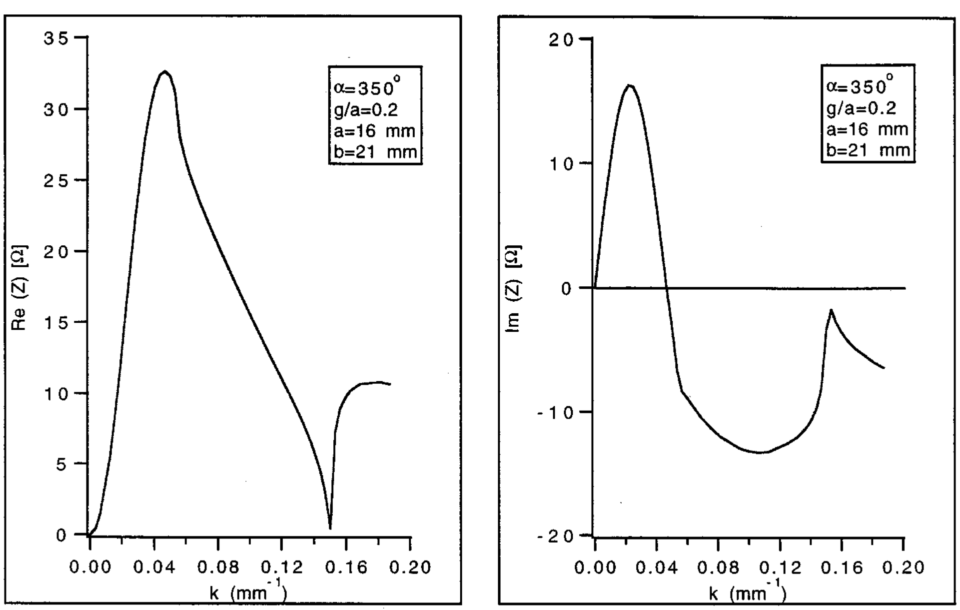

FIG. 6. Real and imaginary parts of the coupling impedance of transverse rectangular slot with the azimuthal length $\alpha \equiv 2B$ equal to 350°.

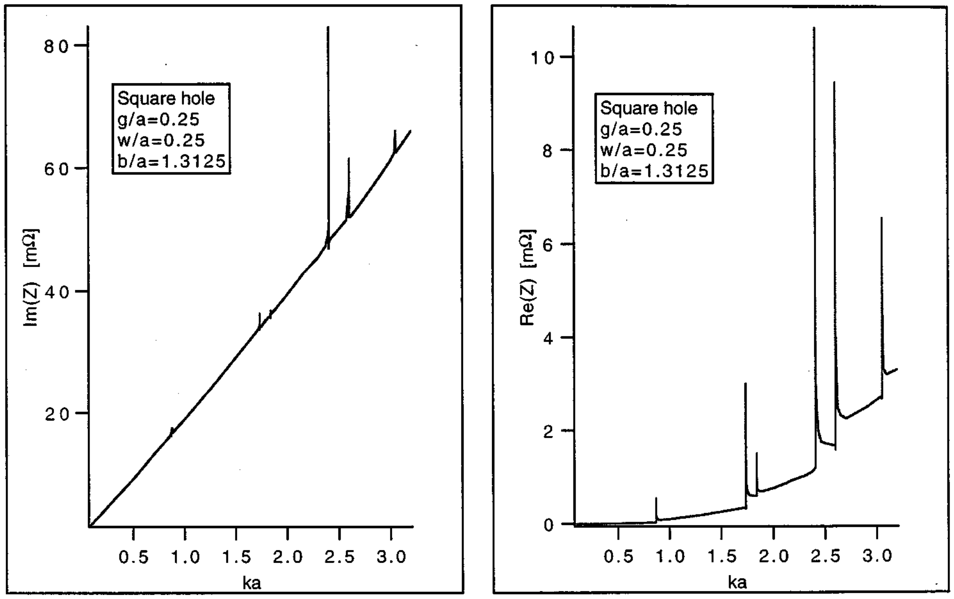

FIG. 7. Frequency dependence of the imaginary and real parts of the impedance for a small square hole (for $a=16$ mm it corresponds to $w=g=4$ mm).

with $Z_0=120\pi$ (units Ω), a the inner pipe radius, c_0 the velocity of light, ω the angular frequency, α_m and α_e the magnetic susceptibility and electric polarizabilty, respectively. For the rectangular hole α_m and α_e are given approximately, for $w/g \leqslant 1$, by [8]:

$$\alpha_m=\frac{\pi}{16}w^2g\left(1+0.3577\,\frac{w}{g}-0.0356\,\frac{w^2}{g^2}\right), \qquad (3.51)$$

$$\alpha_e=\frac{-\pi}{16}w^2g\left(1-0.5663\,\frac{w}{g}+0.1398\,\frac{w^2}{g^2}\right), \qquad (3.52)$$

where w and g are the width and length of the slot, respectively. For the frequency $k_0=1$ GHz and dimensions of the square hole given above, one obtains

$$Z=j0.0073\ \ \Omega. \qquad (3.53)$$

At this point we note that in the problem that we consider, there is in addition the wall of the outer pipe at radius $b=21$ mm, with $b-a=5$ mm. Obviously, with such a geometry the result is expected to be influenced by the outer metallic wall. In the presence of the outer wall, one can imagine an image dipole that creates the field in the coaxial region in the same direction as the field in the pipe region, reducing the coupling impedance.

Therefore, because of the outer wall, we expect the result to be less than the one given by Eq. (3.53), and to approach this result when the distance $b-a$ is increased. The study of the impedance behavior with respect to the distance $b-a$ is given in Table II. It suggests the expected asymptotic increase in the imaginary part of the impedance. The detailed investigation of the effect mentioned above lies outside the scope of the present paper.

The numerical results obtained are in reasonably good agreement with the expected values. Results obtained by Scholz [6] for the frequency 1 GHz are a few percent higher and at $b/a=2$ the numerical value obtained by Scholz is already 10% above the result given by Eq. (3.53).

b. Real part. Recently, the analytic formula for the real part of the impedance of a circular hole in a coaxial structure was derived by Palumbo *et al.* [9]. Comparison of our results with those of Palumbo *et al.* [9], at the frequency $ka=0.3351$ (1 GHz), with $a=16$ mm, is given in Table III. Note that some difference is expected due to the fact that their formula is given for a circular hole, and we approximate our rectangle by a circle of the same area whose radius is $r=\sqrt{16/\pi}$ mm. For low frequencies our numerical results show that the real part of the impedance in a coaxial liner varies as k^2 in contrast to the k^4 behavior for the radiation of a hole into free space. The real part of the impedance also decreasess as $1/\ln(b/a)$. Similar behavior was predicted by Palumbo *et al.* [9]. However, our results in Table III and those of Scholz [6] are significantly larger than those obtained by using the formula derived by Palumbo *et al.* [9].

Based on our analysis we can perform the calculation for low frequencies analytically. For frequencies below all cutoffs, we consider only the TEM mode and obtain

$$\mathrm{Re}\left(\frac{Z}{Z_0}\right)=\frac{k^2}{64\pi^3a^4\,\ln(b/a)}(\psi^2+\chi^2), \qquad (3.54)$$

where $\psi=2\alpha_m$ and $\chi=-2\alpha_e$. The available static approximations for α_m and α_e, as well as the expressions with the frequency corrections [1,10], can be now used to estimate the real part of the impedance for holes of different shape. Note that the value of the impedance obtained in this way will be a few percent higher than the real one due to the fact that expressions for α_m and α_e are given in the literature for a hole in a plane metallic wall (without the outer wall which is present in the coaxial structure). In the discussion following Eq. (3.53) we noted that the fields in and near the hole are modified due to the presence of the outer wall. This can be taken into account by introducing a b/a correction factor in α_m and α_e, but it should only be seen as a vehicle to use Eq. (3.54) when the dimensions of the hole are not negligible with respect to the distance $b-a$. As an example, in Table

TABLE IV. The electric polarizability and magnetic susceptibility of a square hole ($g=w$).

	$b/a=1.2$	$b/a=1.3125$	$b/a=1.5$	$b/a\to\infty$
$-\alpha_e/w^3$	0.0998	0.1010	0.1015	0.1082
α_m/w^3	0.2305	0.2339	0.2351	0.2532

IV we present the b/a dependence of α_m and α_e, based on numerical results for the square hole with the edge length equal to 4 mm, and radius of the liner a equal to 16 mm. For $b/a\to\infty$ the values for α_m and α_e are in good agreement with the results available in the literature for a square hole in a metallic plate. For the circular hole, the above expression becomes

$$\mathrm{Re}\left(\frac{Z}{Z_0}\right)=\frac{5k^2r^6}{36\pi^3a^4\ln(b/a)}, \tag{3.55}$$

where r is the radius of the hole. The expression given by Eq. (3.55) is a factor of 5 larger than the one obtained by Palumbo *et al.* [9]. If we were to multiply the results in the last row of Table III by 5, they would correspond to Eq. (3.55) and would now be in good agreement with the numerical calculations.

IV. SUMMARY

In Sec. II we present the analysis of the calculation of the coupling impedance of an annular cut in a coaxial liner of negligible wall thickness. We obtain equations for calculating the even and odd parts of the impedance, expressed in variational form. The use of the variational method makes numerical study fast and accurate. In order to check our technique, an analytic calculation is performed for low frequencies and compared with the numerical results. The agreement between the analytic and numerical results is very good.

In Sec. III we present a detailed analysis of the calculation of the coupling impedance of a rectangular slot in a coaxial liner of negligible wall thickness over a wide frequency range. We obtain equations for calculating the even and odd parts of the impedance, expressed in variational form. All the integrals in the final expressions given by Eqs. (3.39) and (3.49), are already performed, so that it is only necessary to specify the geometrical parameters of interest. The formulas obtained can be used for numerical study of the coupling impedance of transverse and longitudinal rectangular slots of any size as well as different coaxial configurations. Numerical study for a long narrow transverse slot and a small rectangular hole are presented at different frequencies. The numerical results obtained for both the imaginary and the real part correspond to the expected ones for frequencies below and above cutoff. For the small rectangular hole, the behavior of the imaginary part is not totally consistent with the result of Scholz [6]. The real part of the impedance for the frequencies below cutoff is higher than that expected from the formula obtained by Palumbo *et al.* [9]. A discussion of these differences is given in Sec. III C.

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APPENDIX A: EVALUATION OF INTEGRALS

In this section we present the calculation of the integrals and give the final form of expressions for the quantities $L^{(e)}$, $M^{(e)}$, $N^{(e)}$, $L^{(o)}$, $M^{(o)}$, and $N^{(o)}$. With these expressions, Eqs. (3.39) and (3.49) can be used for direct numerical computation of the even and odd parts of the impedance, respectively.

1. Expressions for the numerical calculations of the even part

The expansion forms in Eqs. (3.31), (3.32) should be based on the specific characteristics of the problem geometry. Using the symmetry argument given in the *Introduction* and the correct boundary conditions at the edges of the hole, we choose

$$E_z^{(e)}(a,\theta,z)=\sum_{\mu\nu} a_{\mu\nu}^{(e)}\cos\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right) \tag{A1}$$

and

$$E_\theta^{(e)}(a,\theta,z)=\sum_{\mu\nu} b_{\mu\nu}^{(e)}\sin\left(\frac{\nu\pi}{2A}z\right)\sin\left(\frac{\mu\pi}{2B}\theta\right), \tag{A2}$$

with μ being odd, and ν being even. In Eqs. (A1)–(A2), $2A\equiv g$ is the length of the hole in the longitudinal direction z, and $2B$ is the angular length of the hole in the azimuthal direction θ. Then the matrices $L^{(e)}$, $M^{(e)}$, $N^{(e)}$ in Eq. (3.37) will be given by the following expressions

$$L^{(e)}=\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz'\cos\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right)\cos\left(\frac{\nu'\pi}{2A}z'\right)\cos\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n\int_{-\infty}^{\infty}dq(\cos qz)(\cos qz')(\cos n\theta)(\cos n\theta')k_{11}, \tag{A3}$$

$$M^{(e)}=\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz'\cos\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right)\sin\left(\frac{\nu'\pi}{2A}z'\right)\sin\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n\int_{-\infty}^{\infty}dq(\cos qz)(\sin qz')(\cos n\theta)(\sin n\theta')k_{12}, \tag{A4}$$

$$N^{(e)}=\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz' \sin\left(\frac{\nu\pi}{2A}z\right)\sin\left(\frac{\mu\pi}{2B}\theta\right)\sin\left(\frac{\nu'\pi}{2A}z'\right)\sin\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n\int_{-\infty}^{\infty}dq(\sin qz)(\sin qz')(\sin n\theta)(\sin n\theta')k_{22}, \tag{A5}$$

where k_{ij} are given by Eqs. (3.23)–(3.25). The matrix $P^{(e)}$ in Eq. (3.36) is given by

$$P^{(e)}=\int_{-B}^{B}d\theta\,\cos\left(\frac{\mu\pi}{2B}\theta\right)\int_{-A}^{A}dz\,\cos\left(\frac{\nu\pi}{2A}z\right)(\cos kz). \tag{A6}$$

After evalution of the integrals we obtain the final form for the quantities $L^{(e)}$, $M^{(e)}$, and $N^{(e)}$, needed to evaluate the matrix $H^{(e)}$ in Eq. (3.39). They are given by the following expressions

$$L_0^{(e)}=\frac{16B^2ka\,\sin(\mu\pi/2)\sin(\mu'\pi/2)}{\mu\mu'\pi^2}\left[-\sum_{s=1}G_{b_{0s}}^{L_{\rm even}}-\frac{1}{2\ln(b/a)}G_k^{L_{\rm even}}+\sum_{s=1}\alpha_{0s}G_{c_{0s}}^{L_{\rm even}}\right],\quad n=0, \tag{A7}$$

$$L_n^{(e)}=\sum_n F^L(\mu,\mu',n,B)ka\Bigg\{\left[n\left(\frac{(b/a)^{2n}}{1-(b/a)^{2n}}\right)G_k^{L_{\rm even}}-\sum_{s=1}G_{b_{ns}}^{L_{\rm even}}+\sum_{s=1}\alpha_{ns}G_{c_{ns}}^{L_{\rm even}}\right]$$
$$+\frac{n^2}{k^2a^2}\Bigg[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}[k^2a^2G_k^{L_{\rm even}}-(b'_{ns})^2G_{b'_{ns}}^{L_{\rm even}}]\frac{1}{k^2a^2-(b'_{ns})^2}$$
$$+\sum_{s=1}\frac{\beta_{ns}}{k^2a^2-(c'_{ns})^2}[k^2a^2G_k^{L_{\rm even}}-(c'_{ns})^2G_{c'_{ns}}^{L_{\rm even}}]\Bigg]\Bigg\},\quad n\neq 0, \tag{A8}$$

$$M_0^{(e)}=N_0^{(e)}=0,\quad n=0, \tag{A9}$$

$$M_n^{(e)}=\sum_n F^M(\mu,\mu',n,B)\left[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}G_{b'_{ns}}^{M_{\rm even}}+\sum_{s=1}\beta_{ns}G_{c'_{ns}}^{M_{\rm even}}\right],\quad n\neq 0, \tag{A10}$$

$$N_n^{(e)}=\sum_n F^N(\mu,\mu',n,B)\left[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}G_{b'_{ns}}^{N_{\rm even}}+\sum_{s=1}\beta_{ns}G_{c'_{ns}}^{N_{\rm even}}\right],\quad n\neq 0, \tag{A11}$$

where explicit expressions for F^L, F^M, F^N, G^L, G^M and G^N are given in Eqs. (A40)–(A42) and (A30)–(A32). The coefficients α_{ns}, β_{ns} and the quantities b_{ns}, c_{ns}, b'_{ns}, c'_{ns} are given in Eqs. (A23)–(A28).

Note that in Eq. (A8) terms with $G_k^{L_{\rm even}}$ cancel each other, which corresponds to the fact that only the mode with $n=0$ can propagate in a coaxial region with the speed of light (TEM mode). Therefore, in order not to lose accuracy, terms with $G_k^{L_{\rm even}}$ $(n\neq 0)$ should be excluded from numerical calculations.

2. Expressions for the numerical calculations of the odd part

For the odd problem we choose the following expansion of the fields:

$$E_z^{(o)}(a,\theta,z)=\sum_{\mu\nu}a_{\mu\nu}^{(o)}\sin\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right) \tag{A12}$$

and

$$E_\theta^{(o)}(a,\theta,z)=\sum_{\mu\nu}b_{\mu\nu}^{(o)}\cos\left(\frac{\nu\pi}{2A}z\right)\sin\left(\frac{\mu\pi}{2B}\theta\right), \tag{A13}$$

with μ and ν being odd. In Eqs. (A12) and (A13), $2A\equiv g$ is the length of the hole in the longitudinal direction z, and $2B$ is the angular length of the hole in the azimuthal direction θ, as was true for the even part. Then the matrices $L^{(o)}$, $M^{(o)}$, $N^{(o)}$ in Eq. (3.48) will be given by the following expressions:

$$L^{(o)}=\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz' \sin\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right)\sin\left(\frac{\nu'\pi}{2A}z'\right)\cos\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n \int_{-\infty}^{\infty}dq(\sin qz)(\sin qz')(\cos n\theta)(\cos n\theta')k_{11}, \tag{A14}$$

$$M^{(o)}=-\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz' \sin\left(\frac{\nu\pi}{2A}z\right)\cos\left(\frac{\mu\pi}{2B}\theta\right)\cos\left(\frac{\nu'\pi}{2A}z'\right)\sin\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n \int_{-\infty}^{\infty}dq(\sin qz)(\cos qz')(\cos n\theta)(\sin n\theta')k_{12}, \tag{A15}$$

$$N^{(o)}=\int_{-B}^{B}\int_{-B}^{B}\int_{-A}^{A}\int_{-A}^{A}d\theta d\theta' dz dz' \cos\left(\frac{\nu\pi}{2A}z\right)\sin\left(\frac{\mu\pi}{2B}\theta\right)\cos\left(\frac{\nu'\pi}{2A}z'\right)\sin\left(\frac{\mu'\pi}{2B}\theta'\right)$$
$$\times\sum_n \int_{-\infty}^{\infty}dq(\cos qz)(\cos qz')(\sin n\theta)(\sin n\theta')k_{22}, \tag{A16}$$

where k_{ij} are given by Eqs. (3.23)–(3.25). The matrix $P^{(o)}$ in Eq. (3.47) is given by

$$P^{(o)}=\int_{-B}^{B}d\theta\, \cos\left(\frac{\mu\pi}{2B}\theta\right)\int_{-A}^{A}dz\, \sin\left(\frac{\nu\pi}{2A}z\right)(\sin kz). \tag{A17}$$

After evaluation of the integrals for the quantities $L^{(o)}$, $M^{(o)}$, and $N^{(o)}$ we obtain

$$L_0^{(o)}=\frac{16B^2ka\sin(\mu\pi/2)\sin(\mu'\pi/2)}{\mu\mu'\pi^2}\left[-\sum_{s=1}G_{b_{0s}}^{L_{\rm odd}}-\frac{1}{2\ln(b/a)}G_k^{L_{\rm odd}}+\sum_{s=1}\alpha_{0s}G_{c_{0s}}^{L_{\rm odd}}\right],\quad n=0, \tag{A18}$$

$$L_n^{(o)}=\sum_n F^L(\mu,\mu',n,B)ka\left\{\left[n\left(\frac{(b/a)^{2n}}{1-(b/a)^{2n}}\right)G_k^{L_{\rm odd}}-\sum_{s=1}G_{b_{ns}}^{L_{\rm odd}}+\sum_{s=1}\alpha_{ns}G_{c_{ns}}^{L_{\rm odd}}\right]\right.$$
$$+\frac{n^2}{k^2a^2}\left[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}[k^2a^2G_k^{L_{\rm odd}}-(b'_{ns})^2G_{b'_{ns}}^{L_{\rm odd}}]\frac{1}{k^2a^2-(b'_{ns})^2}\right.$$
$$\left.\left.+\sum_{s=1}\frac{\beta_{ns}}{k^2a^2-(c'_{ns})^2}[k^2a^2G_k^{L_{\rm odd}}-(c'_{ns})^2G_{c'_{ns}}^{L_{\rm odd}}]\right]\right\},\quad n\neq 0, \tag{A19}$$

$$M_0^{(o)}=N_0^{(o)}=0,\quad n=0, \tag{A20}$$

$$M_n^{(o)}=\sum_n F^M(\mu,\mu',n,B)\left[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}G_{b'_{ns}}^{M_{\rm odd}}+\sum_{s=1}\beta_{ns}G_{c'_{ns}}^{M_{\rm odd}}\right],\quad n\neq 0, \tag{A21}$$

$$N_n^{(o)}=\sum_n F^N(\mu,\mu',n,B)\left[\sum_{s=1}\frac{(p'_{ns})^2}{(p'_{ns})^2-n^2}G_{b'_{ns}}^{N_{\rm odd}}+\sum_{s=1}\beta_{ns}G_{c'_{ns}}^{N_{\rm odd}}\right],\quad n\neq 0, \tag{A22}$$

where explicit expressions for F^L, F^M, F^N, G^L, G^M, and G^N are given in Eqs. (A40)–(A42) and (A33)–(A35). In Eqs. (A7)–(A11) and (A18)–(A22), the coefficients α_{ns} and β_{ns} are given by

$$\alpha_{ns}=\frac{J_n^2(\sigma_{ns}b/a)}{J_n^2(\sigma_{ns}b/a)-J_n^2(\sigma_{ns})}, \tag{A23}$$

$$\beta_{ns}=[J'_n(\sigma'_{ns}b/a)]^2/[[J'_n(\sigma'_{ns}b/a)]^2[n^2/(\sigma'_{ns})^2-1]$$
$$-[J'_n(\sigma'_{ns})]^2[n^2/(\sigma'_{ns}b/a)^2-1]], \tag{A24}$$

and the quantities b_{ns}, c_{ns}, b'_{ns}, c'_{ns} are given by

$$b_{ns}^2=k^2a^2-p_{ns}^2,\quad J_n(p_{ns})=0, \tag{A25}$$

$$c_{ns}^2=k^2a^2-\sigma_{ns}^2,\quad F_n(\sigma_{ns})=0,\quad \sigma_{n0}=0, \tag{A26}$$

$$(b'_{ns})^2=k^2a^2-(p'_{ns})^2,\quad J'_n(p'_{ns})=0, \tag{A27}$$

$$(c'_{ns})^2=k^2a^2-(\sigma'_{ns})^2,\quad G'_n(\sigma'_{ns})=0,\quad \sigma'_{n0}=0. \tag{A28}$$

Note that, as for the even case, terms with $G_k^{L_{\text{odd}}}$ should be excluded from Eq. (A19) from numerical calculations.

3. General definitions

Expressions for the evaluation of the matrices G in Eqs. (A7)–(A11) and (A18)–(A22) are obtained by performing integrals over z, z', and q.

For the even part, for example, we intergrate over z and z' first. The remaining integral over q can be expressed in terms of the following integral:

$$I_1=\int_{-\infty}^{\infty}dq\,\frac{q^2\sin^2 qA}{[(\nu\pi/2A)^2-q^2][(\nu'\pi/2A)^2-q^2]}\,\frac{1}{(q^2a^2-w_s^2)}. \tag{A29}$$

The contour goes below any poles on the negative real q axis and above any poles on the positive real q axis in order to satisfy the outgoing wave boundary condition for the fields generated by the obstacle. For the terms with ν not equal to ν' the only contribution comes from simple poles at $q=\pm w_s/a$. For the terms with ν equal ν' an additional contribution comes from the second order poles at $q=\pm\nu\pi/2A$. The resulting expressions for the elements of the matrices $G^{L_{\text{even}}}$, $G^{N_{\text{even}}}$, and $G^{M_{\text{even}}}$ in Eqs. (A7)–(A11) are the following:

$G^{L_{\text{even}}}$:

$$S^{d0},\quad \nu=\nu'=0,$$

$$S^{d},\quad \nu=\nu'\neq 0, \tag{A30}$$

$$w_s\left(\cos\frac{\nu\pi}{2}\right)\left(\cos\frac{\nu'\pi}{2}\right)S^{\text{nond}},\quad \nu\neq\nu';$$

$G^{M_{\text{even}}}$:

$$\left(\frac{\nu\pi}{g/a}\right)\frac{n}{ka}S^{d},\quad \nu=\nu'\neq 0, \tag{A31}$$

$$w_s\left(\frac{\nu'\pi}{g/a}\right)\frac{n}{ka}\left(\cos\frac{\nu\pi}{2}\right)\left(\cos\frac{\nu'\pi}{2}\right)S^{\text{nond}},\quad \nu\neq\nu';$$

$G^{N_{\text{even}}}$:

$$\frac{1}{ka}S^{Nd},\quad \nu=\nu'\neq 0, \tag{A32}$$

$$\frac{1}{ka}\left(\frac{\nu'\pi}{g/a}\right)\left(\frac{\nu\pi}{g/a}\right)\left(\cos\frac{\nu\pi}{2}\right)\left(\cos\frac{\nu'\pi}{2}\right)\frac{(w_s^2-k^2a^2)}{w_s}S^{\text{nond}},$$

$$\nu\neq\nu';$$

with ν,ν' being even. Here the superscripts d and nond correspond to diagonal and non-diagonal elements.

For the odd part of the impedance, the resulting expressions for the elements of the matrices $G^{L_{\text{odd}}}$, $G^{N_{\text{odd}}}$, and $G^{M_{\text{odd}}}$ in Eqs. (A18)–(A22) are the following:

$G^{L_{\text{odd}}}$:

$$S^{d},\quad \nu=\nu'\neq 0, \tag{A33}$$

$$w_s\left(\sin\frac{\nu\pi}{2}\right)\left(\sin\frac{\nu'\pi}{2}\right)S^{\text{nond}},\quad \nu\neq\nu';$$

$G^{M_{\text{odd}}}$:

$$\left(-\frac{\nu\pi}{g/a}\right)\frac{n}{ka}S^{d},\quad \nu=\nu'\neq 0, \tag{A34}$$

$$-w_s\left(\frac{\nu'\pi}{g/a}\right)\frac{n}{ka}\left(\sin\frac{\nu\pi}{2}\right)\left(\sin\frac{\nu'\pi}{2}\right)S^{\text{nond}},\quad \nu\neq\nu',$$

$G^{N_{\text{odd}}}$:

$$\frac{1}{ka}S^{Nd},\quad \nu=\nu'\neq 0, \tag{A35}$$

$$\frac{1}{ka}\left(\frac{\nu'\pi}{g/a}\right)\left(\frac{\nu\pi}{g/a}\right)\left(\sin\frac{\nu\pi}{2}\right)\left(\sin\frac{\nu'\pi}{2}\right)\frac{(w_s^2-k^2a^2)}{w_s}S^{\text{nond}},$$

$$\nu\neq\nu',$$

with ν,ν' being odd.

In the expressions above we use the following notation

$$S^{d0}=4\pi w_s\frac{1\pm\exp(-jw_s g/a)}{\{[\nu\pi/(g/a)]^2-w_s^2\}^2}+4\pi\frac{jg/a}{[\nu\pi/(g/a)]^2-w_s^2}, \tag{A36}$$

$$S^{d}=4\pi w_s\frac{1\pm\exp(-jw_s g/a)}{\{[\nu\pi/(g/a)]^2-w_s^2\}^2}+2\pi\frac{jg/a}{[\nu\pi/(g/a)]^2-w_s^2}, \tag{A37}$$

$$S^{Nd}=4\pi\left(\frac{\nu\pi}{g/a}\right)^2\frac{(w_s^2-k^2a^2)}{w_s}\,\frac{1\pm\exp(-jw_s g/a)}{\{[\nu\pi/(g/a)]^2-w_s^2\}^2}+2\pi\frac{jg/a\{[\nu\pi/(g/a)]^2-k^2a^2\}}{[\nu\pi/(g/a)]^2-w_s^2}, \tag{A38}$$

$$S^{\text{nond}}=4\pi\frac{1\pm\exp(-jw_s g/a)}{\{[\nu\pi/(g/a)]^2-w_s^2\}\{[\nu'\pi/(g/a)]^2-w_s^2\}}. \tag{A39}$$

In Eqs. (A36)–(A39), the $\pm$ sign corresponds to the even and odd parts, respectively.

Note that, in Eqs. (A30)–(A39), we use general notation w_s, which should be replaced by ka, b_{ns}, c_{ns}, b'_{ns}, or c'_{ns}, as shown by the subscript of the matrix G in Eqs. (A7)–(A11) and (A18)–(A22). Functions F^L, F^M, and F^N in Eqs. (A7)–(A11) and (A18)–(A22) are obtained from the integration over θ and θ'. These functions are given by $F^L(\mu,\mu',n,B)$:

$$B^2, \quad \frac{\mu\pi}{2B}=\frac{\mu'\pi}{2B}=n,$$

$$\frac{2B(\mu'\pi/2B)\sin(\mu'\pi/2)\cos nB}{(\mu'\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}=n\neq\frac{\mu'\pi}{2B}, \tag{A40}$$

$$\frac{2B(\mu\pi/2B)\sin(\mu\pi/2)\cos nB}{(\mu\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}\neq n=\frac{\mu'\pi}{2B}$$

$$\frac{4(\mu\pi/2B)(\mu'\pi/2B)\sin(\mu\pi/2)\sin(\mu'\pi/2)\cos^2 nB}{[(\mu\pi/2B)^2-n^2])[(\mu'\pi/2B)^2-n^2]},$$

$$\frac{\mu\pi}{2B}\neq n\neq\frac{\mu'\pi}{2B};$$

$F^M(\mu,\mu',n,B)$:

$$B^2, \quad \frac{\mu\pi}{2B}=\frac{\mu'\pi}{2B}=n,$$

$$\frac{2Bn\ \sin(\mu'\pi/2)\cos nB}{(\mu'\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}=n\neq\frac{\mu'\pi}{2B},$$

$$\frac{2B(\mu\pi/2B)\sin(\mu\pi/2)\cos nB}{(\mu\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}\neq n=\frac{\mu'\pi}{2B}, \tag{A41}$$

$$\frac{4n(\mu\pi/2B)\sin(\mu\pi/2)\sin(\mu'\pi/2)\cos^2 nB}{[(\mu\pi/2B)^2-n^2][(\mu'\pi/2B)^2-n^2]},$$

$$\frac{\mu\pi}{2B}\neq n\neq\frac{\mu'\pi}{2B};$$

$F^N(\mu,\mu',n,B)$:

$$B^2, \quad \frac{\mu\pi}{2B}=\frac{\mu'\pi}{2B}=n,$$

$$\frac{2Bn\ \sin(\mu'\pi/2)\cos nB}{(\mu'\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}=n\neq\frac{\mu'\pi}{2B}, \tag{A42}$$

$$\frac{2Bn\ \sin(\mu\pi/2)\cos\ nB}{(\mu\pi/2B)^2-n^2}, \quad \frac{\mu\pi}{2B}\neq n=\frac{\mu'\pi}{2B},$$

$$\frac{4n^2\ \sin(\mu\pi/2)\sin(\mu'\pi/2)\cos^2\ nB}{[(\mu\pi/2B)^2-n^2][(\mu'\pi/2B)^2-n^2]}, \quad \frac{\mu\pi}{2B}\neq n\neq\frac{\mu'\pi}{2B}.$$

APPENDIX B: EXPANSION IN TERMS OF SERIES OF ZEROS OF BESSEL FUNCTIONS

In this section we present expressions for the expansion of $P_n(q)$ and $Q_n(q)$, which are present in the integral equation, in terms of algebraic series of zeros of Bessel functions.

1. Expansion of $P_n(q)$

The quantity $P_n(q)$ is given by the following combination of Bessel functions and their derivatives,

$$P_n(q)=\left[\frac{J_n'(\kappa a)}{\kappa a J_n(\kappa a)}-\frac{F_n'(\kappa a)}{\kappa a F_n(\kappa a)}\right], \tag{B1}$$

where $F_n(\kappa a)$ is given by

$$F_n(\kappa a)=Y_n(\kappa a)J_n(\kappa b)-J_n(\kappa a)Y_n(\kappa b), \tag{B2}$$

with $\kappa^2=k^2-q^2$. For the first term we use the relation given in [11]:

$$\Gamma(n+1)(\kappa a/2)^{-n}J_n(\kappa a)=\prod_{s=1}^{\infty}\left(1-\frac{(\kappa a)^2}{p_{ns}^2}\right). \tag{B3}$$

Taking the logarithmic derivatives of both sides, we obtain

$$\frac{J_n'(\kappa a)}{\kappa a J_n(\kappa a)}=\frac{n}{(\kappa a)^2}-2\sum_s \frac{1}{p_{ns}^2-(\kappa a)^2}, \tag{B4}$$

and, using the definition of the propagation constant κ, we obtain

$$\frac{J_n'(\kappa a)}{\kappa a J_n(\kappa a)}=\frac{n}{(\kappa a)^2}-2\sum_s \frac{1}{q^2a^2-b_{ns}^2}. \tag{B5}$$

For the second term in Eq. (B1), the singularity at $\kappa a=\sigma_{ns}$, with $F_n(\sigma_{ns})=0$, gives us

$$\frac{F_n'(\kappa a)}{\kappa a F_n(\kappa a)}=2\sum_{s=1}^{\infty}\frac{\alpha_{ns}}{(\kappa a)^2-\sigma_{ns}^2}, \tag{B6}$$

where α_{ns} is given by the following expression:

$$\alpha_{ns}=\frac{J_n^2(\sigma_{ns}b/a)}{J_n^2(\sigma_{ns}b/a)-J_n^2(\sigma_{ns})}. \tag{B7}$$

There is also a singularity at $\kappa a=0$. For the $n=0$ term, we have the additional terms

$$\frac{F_0'(\kappa a)}{\kappa a F_0(\kappa a)}=-\frac{1}{(\kappa a)^2\ \ln(b/a)}, \tag{B8}$$

and for $n\neq 0$ we have the additional terms

$$\frac{F_n'(\kappa a)}{\kappa a F_n(\kappa a)}=\frac{n}{(\kappa a)^2}\left[\frac{b^{2n}+a^{2n}}{a^{2n}-b^{2n}}\right]. \tag{B9}$$

We can then we rewrite Eq. (B1) as

$$P_n(q)=-\frac{2n}{(\kappa a)^2}\left[\frac{b^{2n}}{a^{2n}-b^{2n}}\right]+\frac{1}{(\kappa a)^2}\frac{1}{\ln(b/a)}\delta_{n0} -2\sum_{s=1}^{\infty}\frac{1}{(qa)^2-b_{ns}^2}+2\sum_{s=1}^{\infty}\frac{\alpha_{ns}}{(qa)^2-c_{ns}^2}, \tag{B10}$$

where

$$b_{ns}^2=k^2a^2-p_{ns}^2, \quad J_n(p_{ns})=0, \tag{B11}$$

$$c_{ns}^2=k^2a^2-\sigma_{ns}^2, \quad F_n(\sigma_{ns})=0, \quad \sigma_{n0}=0. \tag{B12}$$

We rewrite Eq. (B10) as

$$P_n(q)=\begin{cases} -2\sum_{s=1}^{\infty}\frac{1}{q^2a^2-b_{0s}^2}-\frac{1}{\ln(b/a)}\frac{1}{q^2a^2-k^2a^2}+2\sum_{s=1}^{\infty}\frac{\alpha_{0s}}{q^2a^2-c_{0s}^2}, & n=0 \\ -2\sum_{s=1}^{\infty}\frac{1}{q^2a^2-b_{ns}^2}-\frac{b^{2n}}{b^{2n}-a^{2n}}\frac{2n}{q^2a^2-k^2a^2}+2\sum_{s=1}^{\infty}\frac{\alpha_{ns}}{q^2a^2-c_{ns}^2}, & n\neq 0. \end{cases} \tag{B13}$$

At this point we note that after combining expressions for $P_n(q)$ and $Q_n(q)$ in the expression for k_{11} in Eq. (3.23), terms for $n\neq 0$, with singularity at $q=k$, cancel each other. This corresponds to the fact that only the mode with $n=0$ can propagate in the coaxial region with the speed of light (TEM mode).

2. Expansion of $Q_n(q)$

The quantity $Q_n(q)$ is given by the following combination of Bessel functions and its derivatives

$$Q_n(q)=\left[\frac{J_n(\kappa a)}{\kappa a J_n'(\kappa a)}-\frac{G_n(\kappa a)}{\kappa a G_n'(\kappa a)}\right], \tag{B14}$$

where $G_n(\kappa a)$ is given by

$$G_n(\kappa a)=Y_n(\kappa a)J_n'(\kappa b)-Y_n'(\kappa b)J_n(\kappa a). \tag{B15}$$

For the first term in Eq. (B14) one can use the relation given in [11]:

$$2\Gamma(n)(\kappa a/2)^{1-n}J_n'(\kappa a)=\prod_{s=1}^{\infty}\left(1-\frac{(\kappa a)^2}{(p_{ns}')^2}\right). \tag{B16}$$

The final expression for the first term in Eq. (B14) is the following:

$$\frac{J_n(\kappa a)}{\kappa a J_n'(\kappa a)}=-\frac{2}{(\kappa a)^2}\delta_n^0+2\sum_s\left(\frac{(p_{ns}')^2}{(p_{ns}')^2-n^2}\right)\frac{1}{q^2a^2-(b_{ns}')^2}, \tag{B17}$$

where

$$(b_{ns}')^2=k^2a^2-(p_{ns}')^2, \quad J_n'(p_{ns}')=0. \tag{B18}$$

For the second term in Eq. (B14), singularity at $\kappa a=\sigma_{ns}'$, with $G_n'(\sigma_{ns}')=0$, gives us

$$\frac{G_n(\kappa a)}{\kappa a G_n'(\kappa a)}=2\sum_{s=1}^{\infty}\frac{\beta_{ns}}{(\kappa a)^2-(\sigma_{ns}')^2}, \tag{B19}$$

where β_{ns} is given by the following expression:

$$\beta_{ns}=[J_n'(\sigma_{ns}'b/a)]^2/\{[J_n'(\sigma_{ns}'b/a)]^2[n^2/(\sigma_{ns}')^2-1]-[J_n'(\sigma_{ns}')]^2[n^2/(\sigma_{ns}'b/a)^2-1]\}, \tag{B20}$$

In addition, there is a singularity at $\kappa a=0$. For the $n=0$ term, it gives

$$\frac{G_0(\kappa a)}{\kappa a G_0'(\kappa a)}=\frac{2}{(\kappa a)^2[(b^2/a^2)-1]}, \tag{B21}$$

and for $n\neq 0$ there is no additional singularity. Then we rewrite Eq. (B14) as

$$Q_n(q)=-\frac{2}{(\kappa a)^2}\delta_n^0+2\sum_{s=1}^{\infty}\left(\frac{(p_{ns}')^2}{(p_{ns}')^2-n^2}\right)\frac{1}{q^2a^2-(b_{ns}')^2}+2\sum_{s=0}^{\infty}\frac{\beta_{ns}}{(qa)^2-(c_{ns}')^2}, \tag{B22}$$

where b_{ns}' is given by Eq. (B18), c_{ns}' is given by

$$(c_{ns}')^2=k^2a^2-(\sigma_{ns}')^2, \quad G_n'(\sigma_{ns}')=0,$$

$$\sigma_{n0}'=0, \tag{B23}$$

and β_{ns} is given by Eq. (B20) for $s\neq 0$ (for $s=0,n=0$ term it is given by $\beta_{00}=1/[(b^2/a^2)-1]$). We rewrite Eq. (B22) as

$$Q_n(q)=\begin{cases} 2\sum_{s=1}^{\infty}\dfrac{1}{q^2a^2-(b'_{0s})^2}+\dfrac{b^2}{b^2-a^2}\dfrac{2}{q^2a^2-k^2a^2}+2\sum_{s=1}^{\infty}\dfrac{\beta_{0s}}{q^2a^2-(c'_{0s})^2}, & n=0\\ 2\sum_{s=1}^{\infty}\dfrac{1}{q^2a^2-(b'_{ns})^2}\dfrac{(p'_{ns})^2}{(p'_{ns})^2-n^2}+2\sum_{s=1}^{\infty}\dfrac{\beta_{ns}}{q^2a^2-(c'_{ns})^2}, & n\neq 0.\end{cases} \tag{B24}$$

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